APPENDIX. SAMPLE FIELD BEHAVIOR FOR THE FREE MARKOV RANDOM FIELD.

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I. INTRODUCTION

This appendix is concerned with the following question: If μ_0 denotes the Gaussian probability measure on $S'(\mathbb{R}^2)$ with mean zero and covariance $((-\Delta+\mathbb{1})^{-1}f,g)$, what are the properties of "typical" distributions with respect to μ_0? A first result in this direction is given in the final paragraphs of Professor Reed's lectures; he shows that, if $\alpha > 0$, then for almost all $T \in S'(\mathbb{R}^2)$, $\left(\frac{-d^2}{dx_1^2} + \mathbb{1}\right)^{-\alpha} T$ is a locally square-integrable function.

For ease of reference, we will summarize our results here in something less than their full generality:

Theorem 1.1. (a) *The set of distributions* T *having the property that there exists a non-empty open set* U_T *on which* T *is equal to a signed measure is a set of* μ_0*-measure zero.*

*Alfred P. Sloan Foundation Fellow. Also supported in part by NSF Grant GP-15735.

b. *Let* $0 < \alpha \leq \frac{1}{2}$. *The set of distributions* T *such that* $\left(\frac{-d^2}{dx_1^2} + \mathbb{1}\right)^{-\alpha/2} T$ *is a locally Hölder continuous function with all exponents* $\alpha' < \alpha$* *is a set of* μ_0*-measure one.*

c. *Again let* $0 < \alpha \leq \frac{1}{2}$. *The set of distributions* T *such that*

$$\limsup_{x\to\infty} \frac{\left(\frac{-d^2}{dx_1^2} + \mathbb{1}\right)^{-\alpha/2} T(x)}{\sqrt{\log |x|}} = \frac{1}{\pi} \sqrt{\int_{\mathbb{R}} (k^2+1)^{-1}(k_1^2+1)^{-\alpha}\, dk}$$

is a set of μ_0*-measure one.*

Result a) is a negative one; it says that a typical distribution is nowhere sufficiently regular to be a locally integrable function or even a signed measure. Result b) on the other hand says that typical distributions fail to be functions only very slightly. That is, if we regularize so as to add an arbitrarily small fraction of a derivative, typical distributions become Hölder continuous functions. Moreover, it is only necessary to add this fraction of a derivative in one direction, so a typical distribution may be regarded as a continuous function of x_2 with values in the space of distributions in x_1. Thus, although a typical distribution is not a function, it still makes sense to restrict it to a line x_2 = const.

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*We say that a function $T(x)$ is locally Holder continuous with exponent α' if, for each bounded open set $\Lambda \subset \mathbb{R}^2$, there is a constant $Q(\Lambda)$ such that

$$|T(x) - T(y)| \leq Q(\Lambda)|x-y|^{\alpha'}$$

for all $x,y \in \Lambda$.

Result c) gives rather detailed information about the behavior of a typical distribution at infinity. It says in particular that, for almost every T,

$$\left(\frac{-d^2}{dx_1^2} + 1\!\!1\right)^{-\alpha/2} T(x)$$

can be majorized by const. $\times \sqrt{\log |x|}$ for large x but not by any less rapidly growing function of x. Moreover, we will prove that, if ρ is any non-zero element of $S(\mathbb{R}^2)$, then, for almost every T, $T*\rho(x)$ can be majorized by const $\times \sqrt{\log |x|}$ but not by any less rapidly growing function. Thus, we may say that a typical T grows like $\sqrt{\log |x|}$ at infinity. It should be understood, however, that a typical T does not "behave like $\sqrt{\log |x|}$" for large x; instead, it is usually much smaller than $\sqrt{\log |x|}$ and only occasionally takes values which are this large. Indeed, a typical T behaves in some respects as if it were bounded; for example, if $f(x)$ is any square-integrable function, the set of distributions T such that

$$\left(\frac{-d^2}{dx_1^2} + 1\!\!1\right)^{-\alpha/2} T(x)\ f(x)$$

is square-integrable is a set of μ_0-measure one.

Since $S'(\mathbb{R}^n)$ is not a locally compact space, measure theory on it is not quite standard. We will not give a systematic investigation of the subject, but there are a few simple remarks which should make it seem less strange. To begin with, $S'(\mathbb{R}^n)$ comes equipped with three σ-algebras which might in principle be distinct:

Σ_0: the σ-algebra generated by the continuous linear functionals

Σ_w: the σ-algebra generated by the weakly closed sets

Σ_s: the σ-algebra generated by the closed sets.

Evidently, $\Sigma_0 \subset \Sigma_w \subset \Sigma_s$. Gaussian measures are defined on Σ_0, but it is frequently easier to verify that sets we are interested in belong to Σ_s. Fortunately, we have in fact $\Sigma_0 = \Sigma_w = \Sigma_s$. There are general theorems which imply this equality, but in the case at hand it is easy to give a direct elementary proof: By using Hermite expansions, we can identify $S(\mathbb{R}^n)$ with the space s of rapidly decreasing sequences and $S'(\mathbb{R}^n)$ with the space s' of polynomially bounded sequences. Let

$$K_n = \{(a_j) \in s' : |a_j| \leqslant n(1+j^n) \text{ for all } j\} .$$

It is easy to check that K_n is compact and metrizable in the strong topology on s', that $K_n \in \Sigma_0$, and that $s' = \bigcup_{n=1}^{\infty} K_n$. A subset B of s' belongs to $\Sigma_0(\Sigma_s)$ if and only if $B \cap K_n$ belongs to $\Sigma_0(\Sigma_s)$ for all n. Since each K_n is compact, the strong and weak topologies agree on K_n, and since K_n is metrizable, Σ_0 and Σ_w agree on K_n. Hence, Σ_s and Σ_0 agree on K_n for all n so $\Sigma_0 = \Sigma_w = \Sigma_s$. We will from now on denote this σ-algebra simply by Σ, and when we say that a subset of $S'(\mathbb{R}^n)$ is measurable, we mean that it belongs to Σ. We will usually leave the problem of verifying the measurability of the sets we consider to the reader. The fact that $S'(\mathbb{R}^n)$ is a countable union of compact metrizable subsets is frequently useful; to a large extent, it makes possible the reduction of measure theory on $S'(\mathbb{R}^n)$ to measure theory on compact metric spaces.

We will make one small change in the notation used in the Reed lectures. If μ is a Gaussian measure of mean zero on $S'(\mathbb{R}^n)$, the <u>covariance</u> of μ was defined to be the positive semi-definite bilinear form $(.,.)_\mu$ on $S(\mathbb{R}^n)$ given by

$$(f,g)_\mu = \int \mu(dT) \langle f,T\rangle \langle g,T\rangle .$$

By the nuclear theorem, we can write

$$(f,g)_\mu = \int dx\, dy\, f(x)g(y)\, \chi_\mu(x,y),$$

where $\chi_\mu(x,y) = \chi_\mu(y,x)$ is a tempered distribution on $\mathbb{R}^n \times \mathbb{R}^n$. We will also refer to the distribution $\chi_\mu(x,y)$ as the covariance of μ. Furthermore, if μ is translation-invariant,* we can write

$$\chi_\mu(x,y) = \hat{\chi}_\mu(x-y),$$

where $\hat{\chi}_\mu$ is a distribution of positive type on $\mathbb{R}^n$. We will drop the ^ and write, for example,

$$\chi_\mu(x,y) = \chi_\mu(x-y),$$

again referring to $\chi_\mu(\xi)$ as the covariance of μ. In the reverse direction, if we start with a distribution χ of positive type, there is a uniquely determined translation-invariant Gaussian measure with mean zero and covariance $\chi(x-y)$.

We make very limited claims to novelty for the results presented here. Although we have not been able to find these results in the literature, the ideas involved in proving them are quite standard and, indeed, old-fashioned. The general theory of Gaussian stochastic processes has undergone vigorous development in recent years--for a good list of references, see the reviewer's remark in MR 42#4994

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*We are using "translation invariant" as a synonym for the probabilists' term "stationary", i.e., a measure μ on $S'(\mathbb{R}^n)$ is said to be translation invariant if it is invariant under the natural action on $S'(\mathbb{R}^n)$ of the group of translations in $\mathbb{R}^n$.

(1973)--but recent general results do not seem to be directly applicable to the questions investigated here. In any case, our principal objective in writing this appendix was to provide a reasonably self-contained discussion of the answers to these questions in a language accessible to mathematical physicists.

II. REGULARITY.

Let $\alpha > 0$. The mapping

$$T \mapsto \left(\frac{-d^2}{dx_1^2} + \mathbb{1}\right)^{-\alpha/2} T$$

is a continuous linear mapping of $S'(\mathbb{R}^2)$ onto itself. We temporarily denote this mapping by Φ. Schematically, what we want to prove is that

$$\mu_0(\Phi^{-1}\mathcal{H}) = 1$$

where $\mathcal{H}$ is some appropriate space of locally Hölder continuous functions, regarded as a subspace of $S'(\mathbb{R}^2)$. This suggests that we investigate the "support properties" of the measure.

$$\mu = \mu_0 \circ \Phi^{-1} .$$

It is easy to see that μ is again Gaussian with mean zero but has covariance

$$\chi_\mu(x-y) = \frac{1}{(2\pi)^2} \int dp\, e^{ip\cdot x}(p^2+1)^{-1}\,(p_1^2+1)^{-\alpha} \quad . \qquad (2.1)$$

The Fourier transformation can be understood literally, rather than in the sense of distributions, since $(p^2+1)^{-1}(p_1^2+1)^{-\alpha}$ is integrable. This also implies that χ_μ is a continuous function. In fact:

Proposition 2.1 (a). *For* $0 < \alpha < \frac{1}{2}$,

$$|\chi_\mu(0) - \chi_\mu(x)| \leqslant \text{const} \times |x|^{2\alpha}$$

(b) *For* $\alpha = \frac{1}{2}$

$$|\chi_\mu(0) - \chi_\mu(x)| \leqslant \text{const} \times |x| \log\left(\frac{1}{|x|}\right) \quad \textit{for small} \quad |x| .$$

We will return later to the proof of this proposition; assume it for the moment. We are now in the following situation: We have a Gaussian measure μ on $S'(\mathbb{R}^2)$ with mean zero and with a covariance which is Hölder continuous at zero. We want to conclude that μ assigns measure one to the set of locally Hölder continuous functions. We can in fact prove quite a general and precise theorem in this direction; we do not need to assume that the measure is translation invariant, nor are we restricted to a two-dimensional index set. To formulate our result efficiently, we need some notation. Let α, β be real numbers, with $\alpha > 0$. For $0 < \varepsilon < 1$, define

$$\delta_{\alpha,\beta}(\varepsilon) = \varepsilon^{\alpha}\left[\log\left(\frac{1}{\varepsilon}\right)\right]^{-\beta+\frac{1}{2}} . \qquad (2.2)$$

(Note the $\frac{1}{2}$ in the exponent of the logarithm; it is put there for convenience later on.) For any real-valued function T defined on $\mathbb{R}^n$, define

$$\|T\|_{\alpha,\beta} = \sup_{0<|x-y|\leq\frac{1}{2}} \left\{ \frac{|T(x) - T(y)|}{\delta_{\alpha,\beta}(|x-y|)[\log(|x|+2)]^{\frac{1}{2}}} \right\}, \qquad (2.3)$$

and let

$$\mathcal{H}_{\alpha,\beta} = \{T: \|T\|_{\alpha,\beta} < \infty\}.$$

Then $\mathcal{H}_{\alpha,\beta}$ may be identified with a subspace of $S'(\mathbb{R}^n)$ by

$$\langle f,T\rangle = \int f(x)T(x)dx, \qquad f \in S(\mathbb{R}^n).$$

Our principal regularity result is the following:

Theorem 2.2. *Let* μ *be a Gaussian measure on* $S'(\mathbb{R}^n)$ *with mean zero and covariance* $X(x,y)$. *Assume* $X(x,y)$ *is continuous and that for some* $\alpha > 0$, β, *and* c

$$|X(x,x)+X(y,y)-2X(x,y)|^{\frac{1}{2}} \leq c|x-y|^{\alpha}\left[\log\left(\frac{1}{|x-y|}\right)\right]^{-\beta}$$

for all x,y *with* $|x-y| \leq \frac{1}{2}$. *Then*

$$\mu(\mathcal{H}_{\alpha,\beta}) = 1.$$

Note that, by combining Theorem 2.2 with Proposition 2.1 we obtain a sharper version of statement b) of Theorem 1.1, i.e., we show that, for $0 < \alpha < \frac{1}{2}$, μ_0-almost every T has the property that

$$\left(\frac{-d^2}{dx_1^2} + 1\right)^{-\alpha/2} T \quad \text{belongs to} \quad \mathcal{H}_{\alpha,0} \; (\mathcal{H}_{\frac{1}{2},-\frac{1}{2}} \quad \text{for} \quad \alpha = \tfrac{1}{2}).$$

Belonging to $\mathcal{H}_{\alpha,0}$ is a slightly weaker local property than Hölder continuity with exponent α but is stronger than Hölder continuity with all exponents $\alpha' < \alpha$.

In outline, the proof of the theorem goes as follows. Let D^n denote the set of points in $\mathbb{R}^n$ with dyadic rational co-ordinates, and let X denote $\prod_{x \in D^n} \mathbb{R}$, the space of all real-valued functions on D^n. We will again denote elements of X by T, and we will define $\|T\|_{\alpha,\beta}$, for $T \in X$, by the right-hand side of (2.3) with x,y restricted to belong to D^n. Let $\hat{\mu}$ be the Gaussian measure on X with mean zero and covariance $X(x,y)$. By this we mean that $\hat{\mu}$ is the uniquely determined probability measure on X such that, for any $x_1,\ldots,x_m \in D^n$, the random variables $T \mapsto T(x_1),\ldots,T(x_m)$ are jointly Gaussian, and such that

$$\int T(x)\hat{\mu}(dT) = 0; \quad \int T(x)T(y)\hat{\mu}(dT) = X(x,y) \quad \text{for all} \quad x,y \in D^n .$$

<u>Lemma 2.3.</u> *Let* $\tilde{X} = \{T \in X: \|T\|_{\alpha,\beta} < \infty\}$. *Then*

$$\hat{\mu}(\tilde{X}) = 1.$$

Assuming the lemma, we can easily prove the theorem:

Since $\hat{\mu}(\tilde{X}) = 1$, we can restrict the random variables $T(x)$ to $\tilde{X}$ without changing their Gaussian character or their covariance. Also, there is a natural norm-preserving bijection i of $\mathcal{H}_{\alpha,\beta}$ onto $\tilde{X}$ given by

$$i(T) = T|_{D^n} .$$

Let $\hat{\hat{\mu}}$ be the probability measure on $\mathcal{X}_{\alpha,\beta}$ which is the image of $\hat{\mu}$ under i^{-1}. The random variables on $\left(\mathcal{X}_{\alpha,\beta-\frac{1}{2}}, \hat{\hat{\mu}}\right)$ given by

$$T \mapsto T(x) \qquad (x \in D^n)$$

are, by the definition of $\hat{\hat{\mu}}$, jointly Gaussian with mean zero and covariance $X(x,y)$. If $x \in \mathbb{R}^n \backslash D^n$, and if we take a sequence (x_j) in D^n converging to x, we have

$$T(x) = \lim_{j\to\infty} T(x_j)$$

for all $T \in \mathcal{X}_{\alpha,\beta}$. This, together with the continuity of $X(x,y)$, implies that the mappings $T \mapsto T(x)$ are $\hat{\hat{\mu}}$-measurable for all x, and are jointly Gaussian with mean zero and covariance $X(x,y)$. For $f \in S(R^n)$, the mapping

$$T \to \langle f,T \rangle = \int f(x)T(x)dx$$

is again $\hat{\hat{\mu}}$-measurable, Gaussian, and of mean zero, and

$$\int \hat{\hat{\mu}}(dT) \ \langle f_1,T\rangle \langle f_2,T\rangle = \int dx\, dy\, f_1(x)\, f_2(y) X(x,y)$$

(approximate the integral defining $\langle f,T\rangle$ by Riemann sums). Hence, if we regard $\hat{\hat{\mu}}$ as a measure on $S'(\mathbb{R}^n)$ by putting

$$\hat{\hat{\mu}}(S'(\mathbb{R}^n) \backslash \mathcal{X}_{\alpha,\beta}) = 0,$$

$\hat{\hat{\mu}}$ is a Gaussian measure with mean zero and covariance $X(x,y)$. Since a Gaussian measure is uniquely determined by its mean and covariance,

$$\hat{\hat{\mu}} = \mu,$$

and the theorem is proved, once we have proved Lemma 2.3.

Now let $\Delta = [0,1]^n \cap D^n$, and define $\|T\|_{\Delta,\alpha,\beta}$ on X by the right-hand side of (2.3) with x,y restricted to lie in Δ. We next reduce Lemma 2.3 to the following local regularity statement:

<u>Lemma 2.4</u>. *There exist strictly positive constants* c_1, c_2 *depending only on* n, c, α, β *such that, for all positive* λ

$$\hat{\mu}\{T: \|T\|_{\Delta,\alpha,\beta} \geqslant \lambda\} \leqslant c_1 e^{-c_2\lambda^2} .$$

To prove Lemma 2.3, it suffices to show that

$$\lim_{\lambda\to\infty} \hat{\mu}\{T: \sup_{a\in\mathbb{Z}^n} \left\{\|T_{a/2}\|_{\Delta,\alpha,\beta} [\log(|a|+2)]^{-\frac{1}{2}}\right\} \geqslant \lambda\} = 0, \qquad (2.4)$$

where

$$T_{a/2}(x) = T(x-a/2).$$

By Lemma 2.4 and the translation invariance of the hypotheses of Theorem 2.2

$$\hat{\mu}\{T: \|T_{a/2}\|_{\Delta,\alpha,\beta} \geqslant \lambda [\log(|a|+2)]^{\frac{1}{2}}\} \leqslant c_1(|a|+2)^{-c_2\lambda^2}$$

for any $a \in \mathbb{Z}^n$. Hence, the left-hand side of (2.4) is no larger than

$$c_1 \lim_{\lambda\to\infty} \sum_{a\in\mathbb{Z}^n} (|a|+2)^{-c_2\lambda^2} = 0.$$

We now come to the essential and most difficult step--the proof of Lemma 2.4. The argument we will give is a straightforward modification of the argument used by Ito and McKean to construct Wiener measure. (See K. Ito and H.P. McKean, Jr., Diffusion Processes and Their Sample Paths, Springer-Verlag (1965)pp. 12-15.)

For fixed $x,y \in \Delta$, $T(x) - T(y)$ is a Gaussian random variable on $(X,\hat{\mu})$ with mean zero and variance

$$X(x,x) + X(y,y) - 2X(x,y).$$

By the hypotheses of Theorem 2.2, this variance is no larger than $c^2\delta^2_{\alpha,\beta}(|x-y|)\left[\log\left(\frac{1}{|x-y|}\right)\right]^{-1}$ provided that $|x-y| \leqslant \frac{1}{2}$. Hence, again for $0 < |x-y| \leqslant \frac{1}{2}$

$$\hat{\mu}\{T: |T(x)-T(y)| \geqslant \gamma\, \delta_{\alpha,\beta}(|x-y|)\} \leqslant c_3 \cdot |x-y|^{c_4\gamma^2} \qquad (2.5)$$

for all positive γ. Here, c_3, c_4 are strictly positive constants depending only on c. This inequality is the only property of $\hat{\mu}$ we will need to complete the proof.

We now proceed by constructing, for each positive γ, a set $X(\gamma) \subset X$ and proving

a) $\hat{\mu}(X(\gamma)) \leqslant c_1 \exp[-c_5\gamma^2]$ for all positive γ

b) There exists a constant c_6 such that for all γ, all $T \notin X(\gamma)$, and all x,y with $|x-y| \leqslant \frac{1}{2}$,

$$|T(x)-T(y)| \leqslant c_6\, \gamma\, \delta_{\alpha,\beta}(|x-y|)\ .$$

The lemma then follows immediately, with $c_2 = c_5 c_6^{-2}$.

For notational simplicity, we consider only $n = 2$ for the remainder of the argument; the extension to arbitrary n is immediate. We will say that a pair x,y of elements of Δ is an <u>elementary pair (of order</u> j) if

1) the components of x are integral multiples of 2^{-j}

2) each component of y differs from the corresponding component of x either by zero or by $\pm 2^{-j-1}$.

We now define $X(\Upsilon)$ to be the set of $T \in X$ such that, for some $j = 1,2,\ldots$ and some elementary pair (x,y) of order j, $|T(x)-T(y)| > 2\gamma\delta_{\alpha,\beta}(2^{-j-1})$. In other words:

$$\text{If } T \notin X(\gamma), \text{ then } |T(x)-T(y)| \leqslant 2\gamma\delta_{\alpha,\beta}(2^{-j-1}) \text{ for all elementary pairs } x,y \text{ of order } j = 1,2,\ldots . \tag{2.6}$$

To prove a), we pick an elementary pair x,y (order j) and define $z = (x_1,y_2)$. Then $|x-z|$ is 0 or 2^{-j-1}, and similarly for $|y-z|$. Thus

$$\begin{aligned} \hat{\mu}\{|T(x)-T(y)| > 2\gamma\delta_{\alpha,\beta}(2^{-j-1})\} &\leqslant \hat{\mu}\{|T(x)-T(z)| > \gamma\delta_{\alpha,\beta}(2^{-j-1})\} \\ &\quad + \hat{\mu}\{|T(z)-T(y)| > \gamma\delta_{\alpha,\beta}(2^{-j-1})\} \\ &\leqslant 2\, c_3\, 2^{-c_4\gamma^2(j+1)} \end{aligned} \tag{2.7}$$

by (2.5). To estimate $\hat{\mu}(X(\gamma))$, we sum the right-hand side of (2.7) over all elementary pairs. For a given j, the number of elementary pairs of order j is smaller than $8 \times (2^j+1)^2$, so

$$\hat{\mu}(X(\gamma)) \leqslant \sum_{j=1}^{\infty} 8 \times (2^j+1)^2 \cdot 2 \cdot c_3 \cdot 2^{-c_4\gamma^2(j+1)} = O(4^{-c_4\gamma^2}) \text{ as } \gamma \to \infty .$$

Since $\hat{\mu}(X(\gamma)) \leqslant 1$ for all γ, this proves a).

Turning now to the proof of b), we let x, y be two distinct points of Δ with $|x-y| \leqslant \frac{1}{2}$. Let

$$j_0 = \min \{j : \max_{i=1,2} |x_i - y_i| \geqslant 2^{-j}\} .$$

There exist $x^{(0)}$, $y^{(0)}$, whose components are integral multiples of 2^{-j_0}, such that

$$|x_i^{(0)} - x_i| < 2^{-j_0}, \; |y_i^{(0)} - y_i| < 2^{-j_0}, \; |x_i^{(0)} - y_i^{(0)}| \leqslant 2^{-j_0} \; (i=1,2).$$

What we want to show is that, for $T \notin X(\gamma)$,

$$|T(x) - T(y)| \leqslant c_6 \gamma \, \delta_{\alpha,\beta}(|x-y|) .$$

Since $|x-y| \geqslant 2^{-j_0}$, and since $\delta_{\alpha\,\beta}(\varepsilon)$ is increasing in ε for small ε, it suffices to show

$$|T(x) - T(y)| \leqslant c_7 \, \gamma \delta_{\alpha,\beta}(2^{-j_0}). \tag{2.8}$$

By the triangle inequality

$$|T(x)-T(y)| \leqslant |T(x)-T(x^{(0)})| + |T(x^{(0)})-T(y^{(0)})| + |T(y^{(0)})-T(y)|.$$

It follows readily from (2.6) that, if $T \notin X(\gamma)$,

$$|T(x^{(0)})-T(y^{(0)})| \leqslant 4 \, \gamma \delta_{\alpha,\beta}(2^{-j_0-1}).$$

(Put $z^{(0)} = \frac{1}{2}x^{(0)} + \frac{1}{2}y^{(0)}$; then $(x^{(0)}, z^{(0)})$ and $(y^{(0)}, z^{(0)})$ are elementary pairs of order j_0.) The estimates of $|T(x)-T(x^{(0)})|$ and $|T(y)-T(y^{(0)})|$ are identical; we will give only the first of them. Recall that $|x_i-x_i^{(0)}| < 2^{-j_0}$. It is easy to see that we can construct inductively a sequence $x^{(1)}, x^{(2)}, \ldots$ such that

i) $x^{(k)}, x^{(k+1)}$ is an elementary pair of order j_0+k $(k=0,1,2,\ldots)$

ii) $|x_i^{(k)}-x_i| < 2^{-j_0-k}$ $(i=1,2)$.

It follows from ii) (and the fact that each x_i is a dyadic rational) that $x^{(k)} = x$ for sufficiently large k and from i) that, if $T \notin X(\gamma)$

$$|T(x^{(k+1)})-T(x^{(k)})| \leqslant 2\,\gamma\,\delta_{\alpha,\beta}(2^{-j_0-k-1}) \qquad \text{for all } k,$$

so

$$|T(x^{(0)})-T(x)| \leqslant 2\gamma\cdot \sum_{k=0}^{\infty} \delta_{\alpha,\beta}(2^{-j_0-k-1}).$$

Hence, for $T \notin X(\gamma)$,

$$|T(x)-T(y)| \leqslant 4\,\gamma\left[\delta_{\alpha,\beta}(2^{-j_0-1}) + \sum_{k=0}^{\infty} \delta_{\alpha,\beta}(2^{-j_0-k-1})\right].$$

Since $\lim\limits_{j\to\infty} \delta(2^{-j-1})/\delta(2^{-j}) = 2^{-\alpha} < 1$,

the estimate (2.8) follows, completing the proof of Lemma 2.4 and hence of Theorem 2.2.

We note in the following proposition some subsidiary results which follow from the above considerations.

Proposition 2.5 a). *Let* Λ *be a bounded set in* $\mathbb{R}^n$. *There exist constants* c_8, c_9 *(depending on* Λ*) such that*

$$\mu\{T: \sup_{\substack{x,y\in\Lambda \\ |x-y|\leqslant\frac{1}{2}}} \left[\frac{|T(x)-T(y)|}{\delta_{\alpha,\beta}(|x-y|)}\right] \geqslant \lambda\} \leqslant c_8 e^{-c_9\lambda^2}$$

b) *In addition to the hypotheses of Theorem 2.2, assume* $X(x,x)$ *is bounded. Then for* μ*-almost every* T

$$\sup_{x\in\mathbb{R}^n}\left\{\frac{|T(x)|}{\sqrt{\log(|x|+2)}}\right\} < \infty \quad .$$

To prove a), we combine Lemma 2.4 with the argument deriving Theorem 2.2 from Lemma 2.3. To prove b), we note that a) and the fact that $X(x,x)$ is bounded imply the existence of c_{10}, c_{11} such that

$$\mu\left\{T: \sup_{x\in[0,1]^n} \{|T_a(x)|\} \geqslant \lambda\right\} \leqslant c_{10} e^{-c_{11}\lambda^2}$$

for all $a \in \mathbb{Z}^n$; we then argue as in the proof of Lemma 2.3 from Lemma 2.4.

It remains to prove Proposition 2.1. We will consider only $\alpha < \frac{1}{2}$; the proof for $\alpha = \frac{1}{2}$ is similar but slightly messier. We want to estimate:

$$X(0)-X(x) = \frac{1}{(2\pi)^2}\int dp_1 dp_2 \;(p_1^2+p_2^2+1)^{-1}(p_1^2+1)^{-\alpha}\{1-\exp[i(p_1x_1+p_2x_2)]\} \;.$$

Using the fact that the remainder of the integrand is even in p_1 and p_2 separately, we can replace the term in braces by

$$\{1 - \cos(p_1x_1)\cos(p_2x_2)\} \leqslant \{1 - \cos(p_1x_1)\} + \{1 - \cos(p_2x_2)\}.$$

We give the argument for estimating the contribution from the second of the terms on the right; the first is easier. For $\omega > 0$ define

$$\Phi(\omega) = \omega^{1+2\alpha} \int_{-\infty}^{\infty} dp_1 (p_1^2+\omega^2)^{-1}(p_1^2+1)^{-\alpha} = \int_{-\infty}^{\infty} d\sigma (\sigma^2+1)^{-1}(\sigma^2+1/\omega^2)^{-\alpha} ;$$

$\Phi(\omega)$ is continuous for $\omega > 0$ and approaches a finite limit as ω approaches ∞; hence, it is bounded on $[1,\infty)$. Now:

$$\int dp_1 dp_2 (p_1^2+p_2^2+1)^{-1}(p_1^2+1)^{-\alpha}\{1-\cos(p_2x_2)\} =$$

$$= 2\int dp_2 (p_2^2+1)^{-\frac{1}{2}-\alpha} \Phi(\sqrt{p_2^2+1}) \sin^2(p_2x_2/2)$$

$$\leqslant 2\cdot \sup_{\omega\geqslant 1} \Phi(\omega) \cdot x_2^{2\alpha} \cdot \int_{-\infty}^{\infty} d\tau (\tau^2+x_2^2)^{-\frac{1}{2}-\alpha} \sin^2(\tau/2) .$$

Since

$$\int_{-\infty}^{\infty} d\tau (\tau^2+x_2^2)^{-\frac{1}{2}-\alpha} \sin^2(\tau/2)$$

approaches a finite limit as $|x_2|$ approaches zero, we have the desired estimate.

III. NON-REGULARITY.

We will prove in this section an abstract version of statement a) of Theorem 1.1.

Proposition 3.1. *Let* X *be a distribution of positive type on* $\mathbb{R}^n$. *Assume that the Fourier transform of* X *(which must be a positive measure) has infinite total mass; equivalently, assume* X *is not a continuous function. Let* μ *denote the Gaussian measure on* $S'(\mathbb{R}^n)$ *with mean zero and covariance* X. *Then* μ*-almost every distribution*

T has the property that there exists no non-empty open set $U \subset \mathbb{R}^n$ *such that* $T|U$ *is a signed measure.*

Proof. We will need:

Lemma 3.2. *Let* U_1, U_2 *be bounded open sets in* $\mathbb{R}^n$ *with* $\overline{U}_1 \subset U_2$. *Let* ν *be a signed measure of finite total variation on* U_2 *, and let* (φ_n) *be a sequence of continuous functions on* $\mathbb{R}^n$ *with support contained in the open ball of radius* $d(U_1, \mathbb{R}^n \backslash U_2)$ *about* 0, *and with*

$$\sum_m \int |\varphi_m(x)|\,dx < \infty. \tag{3.1}$$

Then

$$\lim_{m\to\infty} \int \nu(dy)\varphi_m(x-y) = 0 \qquad \textit{for almost all} \quad x \in U_1, \tag{3.2}$$

where "almost all" is to be understood in the sense of Lebesgue measure.

Proof:

$$\int_{U_1} dx \left|\int \nu(dy)\varphi_m(x-y)\right| \leqslant |\nu|(U_2)\int |\varphi_m(x)|\cdot dx$$

where $|\nu|$ denotes the total variation of ν. By (3.1) and the Monotone Convergence Theorem,

$$\sum_m \left|\int \nu(dy)\ \varphi_m(x-y)\right| < \infty \quad \text{a.e. on } U_1,$$

which implies (3.2).

Now let $f(x)$ be a C^{∞} function of compact support on $\mathbb{R}^n$ with $\int f(x)dx \neq 0$ and, for each positive λ, define

$$f_{\lambda}(x) = \lambda^n f(\lambda x) .$$

For each fixed x, the mapping

$$T \mapsto (T*f_{\lambda})(x)$$

is a Gaussian random variable on $S'(\mathbb{R}^n)$, with mean zero and variance

$$\int \tilde{X}(dp)|\tilde{f}(p/\lambda)|^2$$

(where $\tilde{X}$ is the Fourier transform of X and $\tilde{f}$ the Fourier transform of f). Since $\tilde{X}$ is a positive measure of infinite total mass, this variance goes to infinity with λ. Choose a sequence of positive numbers a_m with

$$\sum_m a_m < \infty; \qquad (3.3)$$

then a sequence λ_m going to infinity so rapidly that

$$a_m^2 \cdot \int \tilde{X}(dp)| \tilde{f}(p/\lambda_m)|^2 \to \infty ; \qquad (3.4)$$

and put

$$\varphi_m(x) = a_m f_{\lambda_m}(x).$$

It follows from (3.3) and Lemma 3.2 that, if the restriction of T to some non-empty open set is a signed measure, there is a set of positive Lebesgue measure in $\mathbb{R}^n$ on which

$$\lim_{m\to\infty} T*\varphi_m(x) = 0.$$

We will complete the proof of the proposition by showing

$$\text{For almost all } T \in S'(\mathbb{R}^n),\ \limsup_{m\to\infty} |T*\varphi_m(x)| = \infty \text{ a.e.} \tag{3.5}$$

To prove this, we note first that the mapping $(T,x) \mapsto T*\varphi_n(x)$ is continuous and hence Borel from $S'(\mathbb{R}^n) \times \mathbb{R}^n$ to $\mathbb{R}$. Hence,

$$Y \equiv \{(T,x) \in S'(\mathbb{R}^n) \times \mathbb{R}^n\colon \limsup_{m\to\infty} |T*\varphi_m(x)| < \infty\}$$

is a Borel subset of $S'(\mathbb{R}^n) \times \mathbb{R}^n$. Now for any fixed x, the random variables $T \to T*\varphi_m(x)$ $(m=1,2,3,\ldots)$ are Gaussian with variance going to infinity (by (3.4)), so

$$\limsup_{m\to\infty} |T*\varphi_m(x)| = \infty \quad \text{for almost all } T.$$

In other words, for each x,

$$\{T\colon (T,x) \in Y\}$$

is a set of μ-measure zero. By Fubini's Theorem*, Y is a set of $\mu \otimes dx$ measure zero. Applying Fubini's Theorem again, we conclude that, for almost all $T \in S'(\mathbb{R}^n)$, $\{x\colon (T,x) \in Y\}$ is a set of Lebesgue measure zero. This is exactly the desired statement (3.5).

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*There is a slight subtlety at this point. To apply Fubini's Theorem, we need to know that Y belongs to the product σ-algebra, which may in principle by strictly smaller than the σ-algebra of Borel sets in the product space. It is, however, not hard to show, using the fact that $S'(\mathbb{R}^n)$ is a countable union of compact metrizable spaces, that in this case the product σ-algebra and the Borel σ-algebra on the product space actually coincide.

IV. BEHAVIOR AT INFINITY.

Proposition 2.5b implies that the set of distributions which (in some sense) grow no more rapidly than $\sqrt{\log |x|}$ at infinity is a set of μ_0-measure one. In this section, we prove that μ_0-almost every T does in fact grow as fast as $\sqrt{\log |x|}$. It is certainly reasonable to say that a function $T(x)$ grows as fast as $\sqrt{\log |x|}$ at infinity if

$$\limsup_{|x|\to\infty} T(x)[\log(|x|]^{-\frac{1}{2}} > 0.$$

The meaning of the corresponding statement for a distribution is a little more problematical*, but the following two propositions would seem to cover any reasonable interpretation.

Proposition 4.1. *Let* $\alpha > 0$. *Then* μ_0*-almost every* T *satisfies.*

$$\limsup_{|x|\to\infty} \left[\left(\frac{-d^2}{dx_1^2} + \mathbb{1}\right)^{-\alpha/2} T\right](x)[\log(|x|)]^{-\frac{1}{2}} = \frac{1}{\pi}\left[\int dp(p^2+1)^{-1}(p_1^2+1)^{-\alpha}\right]^{\frac{1}{2}}$$

Proposition 4.2. *Let* $\rho \in S(\mathbb{R}^2)$. *Then* μ_0*-almost every* T *satisfies*

$$\limsup_{|x|\to\infty} [T * \check{\rho}(x)\ [\log(|x|)]^{-\frac{1}{2}} = 2[((-\Delta + \mathbb{1})^{-1}\rho,\rho)]^{\frac{1}{2}}.$$

We will prove these propositions by proving the following more general result.

Theorem 4.3. *Let* μ *be a translation-invariant Gaussian measure of mean zero on* $S'(\mathbb{R}^n)$ *with covariance* $X(x-y)$ *which is Hölder continuous at zero and satisfies*

*Note that $2x\cos(x^2)$ is the derivative of the bounded function $\sin(x^2)$. Is the distribution $2x\cos(x^2)$ bounded?

a. $X(0) = 1$

b. $X(x-y)\,|x-y|^n$ is bounded as $|x-y| \to \infty$.

Then

$$\mu\{T: \limsup_{|x|\to\infty} T(x)\,[\log(|x|)]^{-\frac{1}{2}} = \sqrt{2n}\} = 1.$$

To derive the propositions, we apply the theorem to measures of the form

$$\mu = \mu_0 \circ \Phi^{-1}$$

where for Proposition 4.1 we take

$$\Phi(T) = \text{const} \times \left(\frac{-d^2}{dx_1^2} + \mathbb{1}\right)^{-\alpha/2} T$$

and for Proposition 4.2 we take

$$\Phi(T) = \text{const} \times T * \rho \;;$$

the constants are chosen to satisfy the normalization condition a. in Theorem 4.3.

<u>Lemma 4.4.</u> *Let* Λ *be any bounded set in* $\mathbb{R}^n$, *and let* $K_2 < \frac{1}{2}$. *There exists a constant* K_1 *such that*

$$\mu\{T: \sup_{x\in\Lambda} |T(x)| \geqslant \lambda\} \leqslant K_1 \exp[-K_2\lambda^2] \quad \textit{for all positive } \lambda .$$

<u>Lemma 4.5.</u> *For any* $\delta > 0$,

$$\mu\{T: \limsup_{x\to\infty} |T(x)|[\log(|x|)]^{-\frac{1}{2}} > \sqrt{2n(1+\delta)}\} = 0.$$

Lemma 4.6. *For any* $\delta > 0$, *there exists a sequence* x_i *going to* ∞ *in* $\mathbb{R}^n$ *such that*

$$\mu\{T: \limsup_{i} T(x_i)[\log(|x_i|)]^{-\frac{1}{2}} < \sqrt{2n(1-\delta)}\} = 0 \qquad (4.1)$$

The theorem follows at once by applying Lemmas 4.5 and 4.6 to a sequence of positive δ's converging to zero.

Proof of Lemma 4.4. Let $x \in \mathbb{R}^n$. The normalization condition $\chi(0) = 1$ implies

$$\mu\{T: |T(x)| \geqslant \lambda\} \leqslant K_3 \exp[-\lambda^2/2].$$

Hence, for any $x_1,\ldots,x_k$,

$$\mu\{T: \sup_{1\leqslant i\leqslant k} |T(x_i)| \geqslant \lambda\} \leqslant K_3 \cdot k \exp[-\lambda^2/2].$$

By Proposition 2.5a, there exist $\alpha > 0$ and c_8, c_9 such that

$$\mu\{T: |T(x)-T(y)| \leqslant \lambda|x-y|^{\alpha} \text{ for all } x,y \in \Lambda\} \geqslant 1-c_8 \exp[-c_9\lambda^2]. \qquad (4.2)$$

Define ε by

$$(1+\varepsilon)^2 = (2K_2)^{-1}$$

and put $\eta = (2c_9\varepsilon^2)^{\frac{1}{2\alpha}}$. Then it follows from (4.2) that

$$\mu\{T: |T(x)-T(y)| \leqslant \varepsilon\lambda \text{ for all } x,y \text{ with } |x-y| \leqslant \eta\} \geqslant 1-c_8\exp[-\lambda^2/2].$$

If we now choose $x_1,\dots,x_k$ such that each point of Λ is within a distance η of some x_i, we get

$$\mu\{T: \sup_{x\in\Lambda} |T(x)| \geq (1+\varepsilon)\lambda\} \leq (K_3\cdot k+c_8)\exp[-\lambda^2/2],$$

or (replacing $(1+\varepsilon)\lambda$ by λ and recalling the definition of ε)

$$\mu\{T: \sup_{x\in\Lambda}|T(x)| \geq \lambda\} \leq (K_3\cdot k+c_8)\exp[-K_2\lambda^2] .$$

<u>Proof of Lemma 4.5.</u> Now let $\Lambda = [0,1]^n$. Then

$$\mu\{T: \limsup_{|x|\to\infty} \left(|T(x)|[\log(|x|)]^{-\frac{1}{2}}\right) > \sqrt{2n(1+\delta)} =$$

$$\mu\{T: \limsup_{\substack{|a|\to\infty \\ a\in\mathbb{Z}^n}} \left(\sup_{x\in\Lambda}\{|T(x-a)|\}[\log(|a|)]^{-\frac{1}{2}}\right) > \sqrt{2n\ (1+\delta)}\}$$

$$\leq \lim_{A\to\infty} \sum_{\substack{|a|\geq A \\ a\in\mathbb{Z}^n}} \mu\{T: \sup_{x\in\Lambda}\{|T(x-a)|\} \geq \sqrt{2n(1+\delta)\log(|a|)}\} .$$

Now choose $K_2 < \frac{1}{2}$ so that $2K_2(1+\delta) > 1$, and apply Lemma 4.4, (and translation invariance) to get

$$\mu\{T: \sup_{x\in\Lambda} \{|T(x-a)|\} \geq \sqrt{2n(1+\delta)\log(|a|)}\} \leq K_1|a|^{-n'},$$

where $n' = 2K_2(1+\delta)n > n$. Hence

$$\lim_{A\to\infty} \sum_{|a|\geq A} \mu\{T: \sup_{x\in\Lambda} \{|T(x-a)|\} \geq \sqrt{2n(1+\delta)\log(|a|)}\} = 0$$

proving the lemma.

Proof of Lemma 4.6. Let R be a large number (to be chosen later) and let (x_i) be a enumeration of the vectors

$$|a|^{\delta/2}\cdot a, \quad a \in \mathbb{Z}^n, \quad |a| \geqslant R.$$

We choose the enumeration in such a way that $|x_i|$ is increasing in i, and we write ℓ_i for $\log(|x_i|)$ and T_i for the random variable $T \mapsto T(x_i)$. By Lemma 4.5

$$\limsup_{i\to\infty} |T_i|\cdot\ell_i^{-\frac{1}{2}} < \sqrt{3n}$$

for almost all T, so to prove (4.1) it suffices to prove

$$0 = \mu\{T: -\sqrt{3n\ell_i} \leqslant T_i \leqslant \sqrt{2n(1-\delta)\ell_i} \quad \text{for all but finitely many } i\},$$

i.e., for each I

$$0 = \mu\{T: -\sqrt{3n\ell_i} \leqslant T_i \leqslant \sqrt{2n(1-\delta)\ell_i} \quad \text{for all } i \geqslant I\}.$$

We will in fact only prove this equation for $I = 1$ and obtain the result for general I be observing that our argument works for all sufficiently large values of R.

For each $k = 1,2,3,\ldots$ define

$$\pi_k = \mu\{T: -\sqrt{3n\ell_i} \leqslant T_i \leqslant \sqrt{2n(1-\delta)\ell_i} \quad \text{for } 1 \leqslant i \leqslant k\}.$$

What we want to show is that

$$\lim_{k\to\infty} \pi_k = 0.$$

We estimate the π_k's recursively, using the inequality

$$\pi_k \leqslant \pi_{k-1} \cdot \text{ess sup}\left[\mu\{T_k \leqslant \sqrt{2n(1-\delta)\ell_k} \mid T_1 = t_1,\ldots,T_{k-1} = t_{k-1}\}\right]. \quad (4.3)$$

Here, $\mu\{.|..\}$ denotes conditional probability and the essential supremum is taken over all $t_1,\ldots,t_{k-1}$ such that $|t_i| \leqslant \sqrt{3n\ell_k}$ for $1 \leqslant i \leqslant k-1$. (We could, of course, take the essential supremum over a smaller set of t's.) Since the measure μ is Gaussian, it is easy to check that the conditional distribution of T_k given $T_1 = t_1,\ldots,T_{k-1} = t_{k-1}$ may be chosen to be Gaussian with mean

$-\sum_{i=1}^{k-1} \frac{a_{ki}}{a_{kk}} t_i$ and variance a_{kk}^{-1},

where

$$A = (a_{ij})$$

is the $k \times k$ matrix with inverse

$$A^{-1} = B = (b_{ij}) = (X(x_i - x_j)).$$

By condition a) in Theorem 4.3,

$$b_{ii} = 1 \quad \text{for all} \quad i;$$

also, by condition b) and the fact that the x_i are sparsely distributed in $\mathbb{R}^n$, we can make

$$\sum_{\substack{j=1 \\ j \neq i}}^{k} |b_{ij}|$$

small, uniformly in i,k, by making R large.

Now if $C = (c_{ij})$ is any $k \times k$ matrix, the operator norm of

C corresponding to the ℓ^∞ vector norm on $\mathbb{R}^k$ is given by

$$\| C \|_\infty = \sup_{1 \leq i \leq k} \sum_{j=1}^{k} |c_{ij}|$$

Thus, if R is large, $\| B - \mathbb{I} \|_\infty$ is small, so (since $A = B^{-1}$), $\| A - \mathbb{I} \|_\infty$ is also small, so $|a_{kk} - 1|$ and $\sum_{i=1}^{k} \left| \frac{a_{ki}}{a_{kk}} \right|$ are small. We can therefore choose R large enough so that, for all k,

$$\mu\{T_k \leq \sqrt{2n(1-\delta)\ell_k} \;|T_1 = t_1, \ldots, T_{k-1} = t_{k-1}\} \leq 1 - \exp[-\ell_k n(1-\delta/2)]$$

when $|t_i| \leq \sqrt{3n\ell_k}$ for $1 \leq i \leq k-1$. If we do this, and if we use the fact that the ℓ_k's are an enumeration of the numbers $(1+\delta/2)\log(|a|)$, with $a \in \mathbb{Z}^n$ and $|a| \geq R$, we obtain from (4.3)

$$\lim_{k\to\infty} \pi_k \leq \prod_{k=1}^{\infty} \{1 - \exp[-\ell_k n(1-\delta/2)]\} = \prod_{\substack{|a| \geq R \\ a \in \mathbb{Z}^n}} \{1 - |a|^{(1-\delta^2/4)n}\} = 0,$$

completing the proof of the lemma and hence of the theorem.